Generalized parity and quasi-probability density functions

1995 J. Phys. A: Math. Gen. 28 4599

(http://iopscience.iop.org/0305-4470/28/16/017)

J. Phys. A: Math. Gen. 28 (1995) 4599–4608. Printed in the UK

Generalized parity and quasi-probability density functions

M G Benedict and A Czirják
Department of Theoretical Physics, Attila József University, 6701 Szeged POB 428, Hungary

Received 30 January 1995, in final form 19 May 1995

Abstract. By shifting the parity operator in phase space, one obtains a class of operators, which we call the Wigner operator, because its expectation value is equal to the Wigner function. Calculating the commutator of Wigner operators with different arguments, we show that the corresponding observables cannot be measured simultaneously. Introducing a new parameter, we give a trace-class generalization of this operator. The displaced number states of the harmonic oscillator constitute the 'natural' eigenstate basis of the corresponding generalized Wigner operator. Establishing the integral representation of this operator we show that its expectation value is proportional to the Q- and P- and Wigner functions at special values of the parameter. We illustrate this connection with the coherent, number and squeezed states of the harmonic oscillator.

1. Introduction

Generally the state of a classical system is described by a probability density function over the phase space and the physical quantities are represented by functions of the generalized coordinates and momenta, while in the case of a quantum system the corresponding objects are the density operator and the Hermitian operators, respectively.

It is well known that the idea of phase space is problematic in quantum mechanics, since the Heisenberg uncertainty relation forbids the simultaneous characterization of a particle by canonically conjugate variables. For a point particle these variables are the position and the momentum; in the case of an optical mode they are the two quadrature components of the oscillating field. It follows that the state of a quantum system cannot be described by a normal probability density function over the phase space, whose value at a given point would be the probability of finding the system at that point.

However, it is possible to define certain *quasi-probability density functions*, which resemble in many of their properties the classical probability density functions. Thus, we can represent physical quantities by $A(q,p)$ classical functions and physical states by certain $P_Q(p,q)$ quasi-probability density functions which fulfil the condition

$$\langle\hat{A}\rangle = \int \mathrm{d}q \int \mathrm{d}p\, A(q,p) P_Q(q,p) \tag{1}$$

where $\hat{A}$ is the operator corresponding to the classical quantity $A(q,p)$. (For the sake of simplicity, throughout this paper we consider one-dimensional systems, and therefore two-dimensional phase spaces.)

If we require $P_Q(q,p)$ to possess certain properties, detailed in [1, 2], it follows that $P_Q(q,p)$ must be identical to the Wigner function:

$$W_\rho(q,p) = \frac{1}{\pi\hbar}\int \mathrm{d}x\, \langle q-x|\hat{\rho}|q+x\rangle \mathrm{e}^{2\mathrm{i}px/\hbar}. \tag{2}$$

The Wigner function is suitable to represent the state of the system over the corresponding classical phase space if we calculate the expectation value by (1) of a symmetrically ordered operator $\hat{A}$. For complete reviews on the Wigner function see [1–3].

Looking at the definition of the Wigner function, we note that its value in the origin is proportional to the expectation value of the parity operator $\hat{P}_0$:

$$W_\rho(0,0) = \frac{1}{\pi\hbar}\int \mathrm{d}x\ \langle -x|\hat{\rho}|x\rangle = \frac{1}{\pi\hbar}\int \mathrm{d}x\ \langle x|\hat{P}_0\hat{\rho}|x\rangle = \frac{1}{\pi\hbar}\,\mathrm{Tr}(\hat{P}_0\hat{\rho}) \tag{3}$$

and that the proportionality factor consists of universal constants.

In what follows we shall show that the value of the Wigner function at any point of the phase space can be related to the expectation value of an operator in a similar manner, which therefore can be given the name Wigner operator.

We note that this concept has recently been introduced in a different context in [4]. The connection between parity and the Wigner function was already implicitly contained in the work of Cahill and Glauber [5] and later noticed again by Englert [6]. In [7] an experiment has been described to measure the parity of an optical field mode, i.e. the value of the Wigner function at one special point, namely at the origin.

In section 2 we first establish the correspondence between the Wigner operator and the Wigner function by using the variables q and p. Then going over to complex coordinates and calculating the commutator of Wigner operators with different arguments we show that they represent incompatible physical quantities. In section 3 we give a trace-class generalization of this operator and point out that the displaced number states of the harmonic oscillator form the 'natural' eigenstate basis of this generalized Wigner operator. Using its integral representation we show that its expectation value is proportional to other quasi-probability density functions at special values of the parameter. Finally, we illustrate this connection with the coherent, number and squeezed states of the harmonic oscillator.

2. *The shifted parity operator and the Wigner function*

Comparing formulae (2) and (3) we note the phase factor and that the matrix elements of the density operator are to be taken between coordinate basis states displaced relative to each other.

Since the displacement operator

$$\hat{D}(q,p) = \mathrm{e}^{\mathrm{i}(p\hat{q}-q\hat{p})/\hbar} \tag{4}$$

affects the elements of the coordinate basis as

$$\hat{D}(q,p)|x\rangle = \mathrm{e}^{\frac{1}{2}q\,p/\hbar+\mathrm{i}px/\hbar}|x+q\rangle \tag{5}$$

we assume that a Wigner operator can be defined as

$$\hat{W}(q,p) = \frac{1}{\pi\hbar}\hat{D}(q,p)\hat{P}_0\hat{D}^{-1}(q,p) \tag{6}$$

where the factor $1/\pi\hbar$ ensures that we get precise correspondence with the Wigner function in the connection stated in the introduction.

2.1. *Expectation value of the Wigner operator*

We are now going to give a proof for the connection between the Wigner function and the Wigner operator.

Statement. In any state the expectation value of the Wigner operator corresponding to any fixed point in phase space is identical to the value of the Wigner function of that particular state at that specified point:

$$\langle \hat{W}(q,p)\rangle_\rho = W_\rho(q,p). \tag{7}$$

Proof. To verify this relation we represent the physical state by the density operator $\hat{\rho}$ and expand the trace in the coordinate basis. Then

$$\begin{aligned}\langle \hat{W}(q,p)\rangle_\rho &= \mathrm{Tr}(\hat{\rho}\ \hat{W}(q,p)) \\ &= \int \mathrm{d}x\ \langle x|\hat{\rho}\frac{1}{\pi\hbar}\hat{D}(q,p)\hat{P}_0\hat{D}^{-1}(q,p)|x\rangle \\ &= \frac{1}{\pi\hbar}\int \mathrm{d}x \int \mathrm{d}y\ \langle x|\hat{\rho}|y\rangle\langle y|\hat{D}(q,p)\hat{P}_0\hat{D}^{-1}(q,p)|x\rangle.\end{aligned} \tag{8}$$

Using (5) it is easy to obtain the formulae for $\hat{D}^{-1}(q,p)|x\rangle$ and $\langle y|\hat{D}(q,p)$ in the above formula. Since $\hat{P}_0|x-q\rangle = |q-x\rangle$ and $\langle y-q|q-x\rangle = \delta(y-(2q-x))$ we obtain:

$$\begin{aligned}\langle \hat{W}(q,p)\rangle_\rho &= \frac{1}{\pi\hbar}\int \mathrm{d}x \int \mathrm{d}y\ \langle x|\hat{\rho}|y\rangle \mathrm{e}^{\mathrm{i}py/\hbar}\delta(y-(2q-x))\mathrm{e}^{-\mathrm{i}px/\hbar} \\ &= \frac{1}{\pi\hbar}\int \mathrm{d}x\ \langle x|\hat{\rho}|2q-x\rangle \mathrm{e}^{2\mathrm{i}p(q-x)/\hbar} \\ &= \frac{1}{\pi\hbar}\int \mathrm{d}z\ \langle q-z|\hat{\rho}|q+z\rangle \mathrm{e}^{2\mathrm{i}pz/\hbar} \\ &= W_\rho(q,p).\end{aligned} \tag{9}$$

□

In what follows we are going to use the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}\left(\mu\hat{q} + \frac{\mathrm{i}}{\mu}\hat{p}\right) \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\mu\hat{q} - \frac{\mathrm{i}}{\mu}\hat{p}\right) \tag{10}$$

and the corresponding complex coordinates α and α^* in phase space. The parameter μ depends on the particular system under consideration. With this notation the displacement operator and the Wigner operator take the forms

$$\hat{D}(\alpha) = \mathrm{e}^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \tag{11}$$

$$\hat{W}(\alpha) = \frac{1}{\pi\hbar}\hat{D}(\alpha)\hat{P}_0\hat{D}^{-1}(\alpha). \tag{12}$$

It is interesting to note that apart from the factor $1/(\pi\hbar)^2$ the square of the Wigner operator is just the identity operator. Thus the Wigner operator is one of the self-adjoint square roots of the identity. This does not contradict known theorems about the square root of positive self-adjoint operators [8], because the Wigner operator is not positive.

2.2. *Commutator of Wigner operators with different arguments*

The observables represented by the Wigner operators with different arguments cannot be measured simultaneously, because Wigner operators with different arguments do not commute.

Now we calculate this commutator. With the help of the anticommutators

$$\{\hat{P}_0, \hat{a}\} = 0 \qquad \{\hat{P}_0, \hat{a}^\dagger\} = 0 \tag{13}$$

it is easy to verify that $\hat{P}_0\hat{D}(\alpha) = \hat{D}(-\alpha)\hat{P}_0$. Using this equation

$$\begin{aligned}\hat{W}(\alpha)\hat{W}(\beta) &= \frac{1}{(\pi\hbar)^2}\hat{D}(\alpha)\hat{P}_0\hat{D}(-\alpha)\hat{D}(\beta)\hat{P}_0\hat{D}(-\beta)\\ &= \frac{1}{(\pi\hbar)^2}\hat{D}(\alpha)\hat{D}(\alpha)\hat{P}_0\hat{P}_0\hat{D}(-\beta)\hat{D}(-\beta).\end{aligned} \tag{14}$$

Since the product of two displacement operators is

$$\hat{D}(\alpha)\hat{D}(\beta) = e^{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)}\hat{D}(\alpha+\beta) \tag{15}$$

we find that

$$\begin{aligned}\hat{W}(\alpha)\hat{W}(\beta) &= \frac{1}{(\pi\hbar)^2}\hat{D}(2\alpha)\hat{D}(-2\beta)\\ &= \frac{1}{(\pi\hbar)^2}e^{\alpha^*\beta-\alpha\beta^*}\hat{D}(2(\alpha-\beta)).\end{aligned} \tag{16}$$

So the commutator of two Wigner operators is

$$\left[\hat{W}(\alpha), \hat{W}(\beta)\right] = \frac{1}{(\pi\hbar)^2}\left(e^{\alpha^*\beta-\alpha\beta^*}\hat{D}(2(\alpha-\beta)) - e^{\beta^*\alpha-\beta\alpha^*}\hat{D}(2(\beta-\alpha))\right) \tag{17}$$

which equals zero if and only if $\alpha = \beta$.

This result is the consequence of the simple fact that generally a state cannot have a definite parity simultaneously with respect to different points in the phase space.

3. Trace-class generalization of the parity and the Wigner operator

Unfortunately the parity operator, and therefore the Wigner operator, are not trace-class operators:

$$\mathrm{Tr}(\hat{W}(\alpha)) = \frac{1}{\pi\hbar}\,\mathrm{Tr}(\hat{D}(\alpha)\hat{P}_0\hat{D}(-\alpha)) = \frac{1}{\pi\hbar}\,\mathrm{Tr}(\hat{P}_0) = \frac{1}{\pi\hbar}\sum_{n=0}^{\infty}(-1)^n \tag{18}$$

which is clearly not a convergent series, but an oscillatory one.

We can, however, introduce operators, which are trace-class, and which can be brought arbitrarly close to the parity operator by changing a parameter.

The convergence, or better to say the divergence, of the series $\sum_{n=0}^{\infty}(-1)^n$ is a problem dating back to Euler. One of the simplest possibilities for generalizing it to a convergent series is the substitution of -1 by a real number λ, with $|\lambda| < 1$. Then the sum of the series $\sum_{n=0}^{\infty}\lambda^n$ obtained this way is $1/(1-\lambda)$ and this tends to $\frac{1}{2}$ if λ goes to $-1+0$. For simplicity we do not want the sum of the generalized series to depend on the particular value of the parameter λ, but we do want its limit to be $\frac{1}{2}$ as $\lambda \to -1+0$. Thus we have to multiply it by $(1-\lambda)/2$.

So, we would like to find some operators depending on the parameter λ which have the trace $\frac{1}{2}(1-\lambda)\sum_{n=0}^{\infty}\lambda^n$. A possible choice is the generalized parity operator $\hat{P}(\lambda)$ which we define by its effect on the elements of the number state basis as

$$\hat{P}(\lambda)|n\rangle = \frac{1-\lambda}{2}\lambda^n|n\rangle \tag{19}$$

for any $|n\rangle$ and for $|\lambda| < 1$.

The term 'generalized parity' is justified in the interval $-1 < \lambda < 0$ because of the alternating sign of λ^n. Furthermore, this operator reduces the weight of a number state in a pure state as $|\lambda|$ decreases or n increases. In the special case $\lambda = 1$ it projects onto the vacuum state (and multiplies by $\frac{1}{2}$).

With the help of $\hat{P}(\lambda)$ it is straightforward to generalize the Wigner operator as

$$\hat{W}(\alpha, \lambda) = \frac{1}{\pi\hbar}\hat{D}(\alpha)\hat{P}(\lambda)\hat{D}(-\alpha). \tag{20}$$

3.1. Eigenstate basis of the Wigner operator

Any complete set of vectors which have a definite parity with respect to the phase-space point α, is an eigenstate basis of the generalized Wigner operator $\hat{W}(\alpha, \lambda)$ for all $|\lambda| < 1$. One such possibility is the set of the displaced number states of the harmonic oscillator:

$$|n, \alpha\rangle \equiv \hat{D}(\alpha)|n\rangle. \tag{21}$$

As can easily be verified with the help of definition (20), these states are the eigenstates of the corresponding $\hat{W}(\alpha, \lambda)$ with the eigenvalues $[(1 - \lambda)/(2\pi\hbar)]\lambda^n$, for all $|\lambda| < 1$. Furthermore, they are also the eigenstates of the Wigner operator with the eigenvalues $(-1)^n/\pi\hbar$.

In what follows we shall need the expansion of the displaced number states in the number state basis. Since

$$\hat{D}(\alpha) = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha\hat{a}^\dagger}\mathrm{e}^{-\alpha^*\hat{a}} \tag{22}$$

and

$$\hat{a}^j|n\rangle = \begin{cases} \sqrt{\dfrac{n!}{(n-j)!}}|n-j\rangle & \text{if } j \leqslant n \\ 0 & \text{if } j > n \end{cases} \tag{23}$$

$$(\hat{a}^\dagger)^k|n\rangle = \sqrt{\frac{(n+k)!}{n!}}|n+k\rangle \tag{24}$$

we can write $|n, \alpha\rangle$ as follows:

$$|n, \alpha\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\sum_{j=0}^{n}\sum_{k=0}^{\infty}\frac{\sqrt{n!(n-j+k)!}}{j!k!(n-j)!}\alpha^k(-\alpha^*)^j|n-j+k\rangle. \tag{25}$$

The scalar product $\langle m|n, \alpha\rangle$ can be expressed with the associated Laguerre polynomials $L_n^{(a)}(x)$ [9]:

$$\langle m|n, \alpha\rangle = \begin{cases} \mathrm{e}^{-|\alpha|^2/2}(-\alpha^*)^{n-m}\sqrt{\dfrac{m!}{n!}}L_m^{(n-m)}(|\alpha|^2) & \text{if } m < n \\ \mathrm{e}^{-|\alpha|^2/2}\alpha^{m-n}\sqrt{\dfrac{n!}{m!}}L_n^{(m-n)}(|\alpha|^2) & \text{if } m \geqslant n. \end{cases} \tag{26}$$

Thus equation (25) can be given another form:

$$|n, \alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\Bigg(\sum_{m=0}^{n}\sqrt{\frac{m!}{n!}}(-\alpha^*)^{n-m}L_m^{(n-m)}(|\alpha|^2)|m\rangle \tag{27}$$

$$+ \sum_{m=n+1}^{\infty}\sqrt{\frac{n!}{m!}}\alpha^{m-n}L_n^{(m-n)}(|\alpha|^2)|m\rangle\Bigg). \tag{28}$$

3.2. Moments of the generalized Wigner operator

Now we are going to calculate the expectation value of the generalized Wigner operator in the state described by the density operator $\hat{\rho}$. Since the expansion of the trace in the suitable displaced number state basis is particularly simple, we may say that this is the 'natural' basis of the generalized Wigner operator:

$$\langle \hat{W}(\alpha,\lambda)\rangle_\rho = \mathrm{Tr}(\hat{\rho}\hat{W}(\alpha,\lambda)) = \sum_{n=0}^{\infty}\langle n,\alpha|\hat{\rho}\hat{W}(\alpha,\lambda)|n,\alpha\rangle$$
$$= \frac{1-\lambda}{2\pi\hbar}\sum_{n=0}^{\infty}\langle n,\alpha|\hat{\rho}|n,\alpha\rangle\lambda^n. \tag{29}$$

We shall use this result in section 6 in specific examples.

It can similarly be obtained that the higher moments of $\hat{W}(\alpha,\lambda)$ are

$$\langle \hat{W}^k(\alpha,\lambda)\rangle_\rho = \left(\frac{1-\lambda}{2\pi\hbar}\right)^k \sum_{n=0}^{\infty}\langle n,\alpha|\hat{\rho}|n,\alpha\rangle\lambda^{kn} \tag{30}$$

for every positive integer k, so one can easily calculate also the variance of the generalized Wigner operator.

4. Integral representation for the generalized Wigner operator

Due to the trace-class extension introduced in the previous section, the Hilbert–Schmidt norm (defined for an arbitrary operator $\hat{F}$ as $\mathrm{Tr}(\hat{F}^\dagger\hat{F})$) of the generalized Wigner operator is finite, namely

$$\mathrm{Tr}(\hat{W}^\dagger(\alpha,\lambda)\hat{W}(\alpha,\lambda)) = \sum_{n=0}^{\infty}\langle n,\alpha|\hat{W}^\dagger(\alpha,\lambda)\hat{W}(\alpha,\lambda)|n,\alpha\rangle$$
$$= \left(\frac{1-\lambda}{2\pi\hbar}\right)^2\sum_{n=0}^{\infty}\lambda^{2n} = \frac{1-\lambda}{4\pi\hbar(1+\lambda)}. \tag{31}$$

Using the special features of the displacement operators, namely that they play the role of a basis in the space of operators with finite Hilbert–Schmidt norm, similarly to the Fourier basis in the space of the square-integrable functions, Cahill and Glauber have shown [5] that any operator $\hat{F}$ with a finite Hilbert–Schmidt norm can be written in the form

$$\hat{F} = \frac{1}{\pi}\int \mathrm{d}^2\xi\; f(\xi)\hat{D}(\xi) \tag{32}$$

where

$$f(\xi) = \mathrm{Tr}(\hat{F}\hat{D}^{-1}(\xi)) \tag{33}$$

is square-integrable. The validity of (32) can be easily seen using the 'orthogonality relation'

$$\mathrm{Tr}(\hat{D}(\xi)\hat{D}^{-1}(\xi')) = \pi\delta(\xi-\xi') \tag{34}$$

which can be obtained with the help of (15).

If we apply equation (33) to the generalized Wigner operator and expand the trace on the displaced number states (21), we find that

$$w(\xi,\alpha,\lambda) \equiv \mathrm{Tr}(\hat{D}^{-1}(\xi)\hat{W}(\alpha,\lambda)) = \sum_{n=0}^{\infty}\langle n,\alpha|\hat{D}^{-1}(\xi)\hat{W}(\alpha,\lambda)|n,\alpha\rangle = \frac{1-\lambda}{2\pi\hbar}\sum_{n=0}^{\infty}\langle n,\alpha|\hat{D}(-\xi)|n,\alpha\rangle\lambda^{n}. \tag{35}$$

Let us have a closer look at the matrix element in the above sum. If we apply equations (21) and (15) we get

$$\langle n,\alpha|\hat{D}(-\xi)|n,\alpha\rangle = \mathrm{e}^{\alpha\xi^{*}-\alpha^{*}\xi}\langle n|\hat{D}(-\xi)|n\rangle. \tag{36}$$

Using equation (26) in the special case of $m = n$, the matrix element of the displacement operator between the number states above takes the form

$$\langle n|\hat{D}(-\xi)|n\rangle = \mathrm{e}^{-|\xi|^2/2}L_n^{(0)}(|\xi|^2). \tag{37}$$

If we back-substitute equations (36) and (37) in (35), we get

$$w(\xi,\alpha,\lambda) = \frac{1-\lambda}{2\pi\hbar}\mathrm{e}^{\alpha\xi^{*}-\alpha^{*}\xi}\mathrm{e}^{-|\xi|^2/2}\sum_{n=0}^{\infty}L_n^{(0)}(|\xi|^2)\lambda^{n}. \tag{38}$$

The sum in the above expression can be written in a closed form (cf 8.975,1. of [9]). This way we arrive at

$$w(\xi,\alpha,\lambda) = \frac{1}{2\pi\hbar}\mathrm{e}^{\alpha\xi^{*}-\alpha^{*}\xi}\exp\left(\frac{1}{2}|\xi|^2\frac{\lambda+1}{\lambda-1}\right). \tag{39}$$

With this 'coefficent function' we can write the integral representation of the generalized Wigner operator as

$$\hat{W}(\alpha,\lambda) = \frac{1}{2\pi^2\hbar}\int \mathrm{d}^2\xi\ \hat{D}(\xi)\exp\left(\frac{1}{2}\frac{\lambda+1}{\lambda-1}|\xi|^2+\alpha\xi^{*}-\alpha^{*}\xi\right). \tag{40}$$

If we take the limit $\lambda \to -1+0$ in (40), we can write the Wigner operator in an integral form:

$$\hat{W}(\alpha) = \frac{1}{2\pi^2\hbar}\int \mathrm{d}^2\xi\ \hat{D}(\xi)\mathrm{e}^{\alpha\xi^{*}-\alpha^{*}\xi} \tag{41}$$

which is essentially the Fourier transform of the displacement operator over the phase space.

5. Connection with other quasi-probability density functions

If we evaluate the expectation value of the generalized Wigner operator using its integral representation (40), we obtain the following:

$$\langle\hat{W}(\alpha,\lambda)\rangle_{\rho} = \frac{1}{2\pi^2\hbar}\int \mathrm{d}^2\xi\ \mathrm{Tr}(\hat{\rho}\hat{D}(\xi))\exp\left(\frac{1}{2}\frac{\lambda+1}{\lambda-1}|\xi|^2+\alpha\xi^{*}-\alpha^{*}\xi\right). \tag{42}$$

This is simply the s-ordered quasi-probability density function of Cahill and Glauber [5] with the correspondence $s = (\lambda+1)/(\lambda-1)$, which interpolates between the Wigner, Q- and P-functions. Thus the generalized Wigner operator is essentially a *quasi-probability density operator*, since its expectation value is the whole class of quasi-probability density functions along the parameter λ or s, equivalently.

We note that the interval $-1 < \lambda \leqslant 0$, where the generalized parity operator transforms the states with an alternating sign, corresponds to the well-behaved quasi-probability density functions.

In what follows we detail the connection of the generalized Wigner operator with the Q- and P-functions.

5.1. The Q-function

In the case of the Q-function we can also obtain the correspondence with the help of (29). If we write $\lambda = 0$ in the expression of $\hat{W}(\alpha, \lambda)$ we get an operator which projects every state onto a coherent state (and multiplies it by $1/2\pi\hbar$). In accord with this, if we substitute $\lambda = 0$ in (29), there remains only the zeroth term in the sum, while all the others vanish:

$$\langle \hat{W}(\alpha, \lambda = 0)\rangle_\rho = \frac{1}{2\pi\hbar}\langle 0|\hat{D}(-\alpha)\hat{\rho}\hat{D}(\alpha)|0\rangle = \frac{1}{2\pi\hbar}\langle \alpha|\hat{\rho}|\alpha\rangle = \frac{1}{2\hbar}Q(\alpha). \tag{43}$$

Thus the expectation value of the generalized Wigner operator at $\lambda = 0$ is proportional to the Q-function.

5.2. The P-function

Since the P-function is singular in general [10], we cannot expect that in the convergence interval of (29) for λ we get the P-function.

However, if we let $\lambda \to -\infty$, apart from the factor $1/2\hbar$ we obtain the P-function of the corresponding state.

Taking into account (22) the trace in (42) can be written as

$$\mathrm{Tr}(\hat{\rho}\hat{D}(\xi)) = \mathrm{e}^{-|\xi|^2/2}\,\mathrm{Tr}(\hat{\rho}\mathrm{e}^{\xi\hat{a}^\dagger}\mathrm{e}^{-\xi^*\hat{a}}) = \mathrm{e}^{-|\xi|^2/2}\chi_{\mathrm{N}}(\xi). \tag{44}$$

where $\chi_{\mathrm{N}}(\xi)$ is the characteristic function for normal ordering [11]. Thus an equivalent form of (42) is the following:

$$\langle \hat{W}(\alpha, \lambda)\rangle_\rho = \frac{1}{2\pi^2\hbar}\int \mathrm{d}^2\xi\ \chi_{\mathrm{N}}(\xi)\exp\left(\frac{1}{\lambda - 1}|\xi|^2 + \alpha\xi^* - \alpha^*\xi\right). \tag{45}$$

Now in (45) the first term of the exponent vanishes as $\lambda \to -\infty$, so there remains only the Fourier transform of the characteristic function for normal ordering, which by definition is the P-function [11].

6. Examples

Finally we illustrate the above connections by calculating the expectation value of the generalized Wigner operator in the coherent, number and squeezed states of the harmonic oscillator, using formula (29).

6.1. Coherent state

Let the state of an oscillator be the coherent state $|\beta\rangle$. In this case the density operator is $\hat{\rho}_\beta = |\beta\rangle\langle\beta|$, so the matrix element to be calculated in (29) is the following:

$$\langle n, \alpha|\hat{\rho}_\beta|n, \alpha\rangle = |\langle\beta|n, \alpha\rangle|^2 = \mathrm{e}^{-|\alpha-\beta|^2}\frac{|\alpha - \beta|^{2n}}{n!}. \tag{46}$$

After substitution into (29) we get

$$\langle \hat{W}(\alpha, \lambda)\rangle_\beta = \frac{1-\lambda}{2\pi\hbar} \mathrm{e}^{|\alpha-\beta|^2(\lambda-1)}. \tag{47}$$

It can readily be seen that in the limits of λ at 0 and -1 $\langle \hat{W}(\alpha, \lambda)\rangle_\beta$ is the well known Q- and Wigner function of the coherent state, respectively, while as $\lambda \to -\infty$, $\langle \hat{W}(\alpha, \lambda)\rangle_\beta$ tends to $\delta(\alpha - \beta)$ which is the P-function of the coherent state multiplied by $1/2\hbar$.

6.2. Number state

In the case of a number state $\hat{\rho}_m = |m\rangle\langle m|$, thus the matrix element in (29) is

$$\langle n, \alpha|\hat{\rho}_m|n, \alpha\rangle = |\langle m|n, \alpha\rangle|^2. \tag{48}$$

Using (26) and back-substituting in (29), we get the following result for the expectation value of the generalized Wigner operator in a number state:

$$\langle \hat{W}(\alpha, \lambda)\rangle_m = \frac{1-\lambda}{2\pi\hbar} \mathrm{e}^{-|\alpha|^2} \left\{ \sum_{n=0}^{m} \lambda^n |\alpha|^{2(m-n)} \frac{n!}{m!} \left(L_n^{(m-n)}(|\alpha|^2)\right)^2 \right. \tag{49}$$

$$\left. + \sum_{n=m+1}^{\infty} \lambda^n |\alpha|^{2(n-m)} \frac{m!}{n!} \left(L_m^{(n-m)}(|\alpha|^2)\right)^2 \right\}. \tag{50}$$

It can be shown that for $\lambda \to -1+0$ this formula reproduces the Wigner function of the number state $|m\rangle$:

$$W_m(\alpha) = \frac{(-1)^m}{\pi\hbar} \mathrm{e}^{-2|\alpha|^2} L_m^{(0)}(4|\alpha|^2). \tag{51}$$

As a byproduct of this calculation we have obtained the following non-trivial relation between the Laguerre polynomials:

$$L_m^{(0)}(4x) = \mathrm{e}^x \left\{ \sum_{n=0}^{m} (-x)^{(m-n)} \frac{n!}{m!} \left(L_n^{(m-n)}(x)\right)^2 + \sum_{n=m+1}^{\infty} (-x)^{(n-m)} \frac{m!}{n!} \left(L_m^{(n-m)}(x)\right)^2 \right\}. \tag{52}$$

6.3. Squeezed state

If the oscillator is in a squeezed coherent state, it can be described by the ket

$$|\beta, \xi\rangle \equiv \hat{D}(\beta)\hat{S}(\xi)|0\rangle \tag{53}$$

where $|0\rangle$ is the vacuum state of the harmonic oscillator, and

$$\hat{S}(\xi) = \mathrm{e}^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi(\hat{a}^\dagger)^2)} \tag{54}$$

is the squeezing operator. The matrix element

$$\langle n, \alpha|\hat{\rho}_{\beta,\xi}|n, \alpha\rangle = |\langle n|\hat{D}(-\alpha)\hat{D}(\beta)\hat{S}(\xi)|0\rangle|^2 = |c_n(\beta - \alpha)|^2 \tag{55}$$

where $c_n(\beta - \alpha)$ is the expansion coefficient of the $|\beta - \alpha, \xi\rangle$ squeezed state in the number state basis. These coefficients can be written in the analytic form [12]:

$$c_n(\beta - \alpha) = \exp\left(-\tfrac{1}{2}|\beta - \alpha|^2 - \tfrac{1}{2}(\beta^* - \alpha^*)^2 \mathrm{e}^{\mathrm{i}\Theta} \tanh(r)\right) \sqrt{\frac{(\frac{1}{2}\mathrm{e}^{\mathrm{i}\Theta} \tanh(r))^n}{n! \cosh(r)}}$$

$$\times H_n\left(\frac{(\beta - \alpha)\cosh(r) + (\beta^* - \alpha^*)\mathrm{e}^{\mathrm{i}\Theta} \sinh(r)}{\sqrt{\mathrm{e}^{\mathrm{i}\Theta} \sinh(2r)}}\right) \tag{56}$$

where $re^{i\Theta} = \xi$ and H_n is the nth Hermite polynomial. So the expectation value of the generalized Wigner operator in the squeezed state (53) is

$$\langle\hat{W}(\alpha,\lambda)\rangle_{\beta,\xi} = \frac{1-\lambda}{2\pi\hbar}\exp\left(-|\beta-\alpha|^2 - \tfrac{1}{2}\left((\beta^*-\alpha^*)^2 e^{i\Theta} + (\beta-\alpha)^2 e^{-i\Theta}\right)\tanh(r)\right)$$
$$\times\sum_{n=0}^{\infty}\frac{(\frac{1}{2}e^{i\Theta}\tanh(r)\lambda)^n}{n!\cosh(r)}\left|H_n\left(\frac{(\beta-\alpha)\cosh(r)+(\beta^*-\alpha^*)e^{i\Theta}\sinh(r)}{\sqrt{e^{i\Theta}\sinh(2r)}}\right)\right|^2. \tag{57}$$

Acknowledgments

This work has been supported by the National Research Fund of Hungary (OTKA) No 1977.

References

[1] Hillery M O'Connel R F Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121
[2] O'Connel R F and Wigner E P 1981 *Phys. Lett.* **83A** 145
[3] Kim Y S and Noz M E 1991 *Phase Space Picture of Quantum Mechanics* (Singapore: World Scientific)
[4] Yeh L and Kim Y S 1994 *Found. Phys.* **24** 873
[5] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1857, 1882
[6] Englert B-G 1989 *J. Phys. A: Math. Gen.* **22** 625
[7] Englert B-G, Sterpi N and Walther H 1993 *Opt. Commun.* **100** 526
[8] Riesz F and B Sz-Nagy 1955 *Functional Analysis* (New York: Ungar)
[9] Gradshteyn I S and Ryzhik I M 1980 *Tables of Integrals Series and Products* (New York: Academic)
[10] Hillery M 1985 *Phys. Lett.* **111A** 409
[11] Gardiner C W 1991 *Quantum Noise* (Berlin: Springer)
[12] Gong J J and Aravind P K 1990 *Am. J. Phys.* **58** 1003